Generalized Fiducial Inference on Orbital Parameters and Exoplanet Detection

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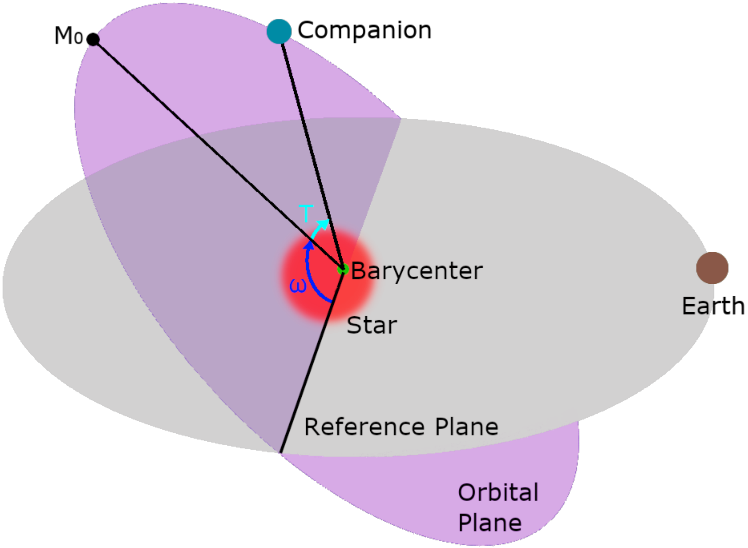
Talk Goals

- Highlight current challenges in estimating planetary orbital parameters
- Propose Generalized Fiducial Inference as potential alternative
- Compare Generalized Fiducial Inference and Bayes performance

Planetary Orbits



Keplerian Orbit Model

$$Y(t) = v(t) + \sqrt{\sigma^2 + \sigma_t^2} Z_t$$

$$v(t) = \sum_{j=1}^{J} C_j \mathbb{1}(\gamma(t) = j) + \sum_{i=1}^{N} K_i [\cos(\omega_i + T_i(t)) + \zeta_i \cos(\omega_i)]$$

- $Z_t \overset{iid}{\sim} N(0,1)$
- J number of spectrographs
- N number of planets
- $T_i(t) = f(t, \zeta_i, P_i, M_{0,i})$
- Observed Data
- Unknown Parameters

E.g Danby (1992); Ford & Gregory (2006); Nelson et al. (2020)

Limitations of Current Methods

- Maximum likelihood estimates cannot incorporate prior information
- Bayesian priors embellish on the known information
- Bayesian model selection sensitive to prior specification (Hannig et al., 2016; Parviainen, 2017)

Generalized Fiducial Inference

- Modern take on Fisher's fiducial argument
- Produces "posterior" distribution for parameter without a prior
- Generalized Fiducial Distribution advantages (Hannig et al., 2016):
  - Never improper
  - Invariant to smooth reparameterizations of the parameters
  - Bounds on parameters can be incorporated through reparameterizations

Generalized Fiducial Inference

Assume a data generating model for data vector $\mathbf{Y}$:

$$\mathbf{Y} = G(\mathbf{Z}, \boldsymbol{\theta})$$

where

- G is a deterministic function
- $\boldsymbol{\theta}$ are the unknown parameters of interest
- $\mathbf{Z}$ has a completely known and fully specified distribution

Fiducial Flip

Similar to relationship between density $f(\mathbf{y}|\boldsymbol{\theta})$ and likelihood $\mathcal{L}(\boldsymbol{\theta})$

0. Recall $\mathbf{Y} = G(\mathbf{Z}, \boldsymbol{\theta})$
1. Observed $\mathbf{y}$ corresponds to realization $\mathbf{z}$ of $\mathbf{Z}$
2. Relationship now implicit and can solve for $\mathbf{z}$
3. Evaluate likelihood of $\mathbf{Z}$ taking on value $G^{-1}(\mathbf{y}, \boldsymbol{\theta})$

Generalized Fiducial Distribution (GFD)

- Under certain regularity conditions, the distributional statistic for $\boldsymbol{\theta}$ is (Hannig et al., 2016):

$$r_{\mathbf{Y}}(\boldsymbol{\theta}) = \frac{\mathcal{L}(\boldsymbol{\theta}) J(\mathbf{Y}, \boldsymbol{\theta})}{\int_{\mathbb{T}} \mathcal{L}(\boldsymbol{\vartheta}) J(\mathbf{Y}, \boldsymbol{\vartheta}) d\boldsymbol{\vartheta}}$$

- $\mathcal{L}(\boldsymbol{\theta})$ is the likelihood
- $J(\mathbf{Y}, \boldsymbol{\theta}) = D(\nabla_{\boldsymbol{\theta}} G(\mathbf{Z}, \boldsymbol{\theta})|_{\mathbf{Z}=G^{-1}(\mathbf{Y},\boldsymbol{\theta})})$
- $D(M) = \sqrt{\det(M'M)}$

Generalized Fiducial Inference Example

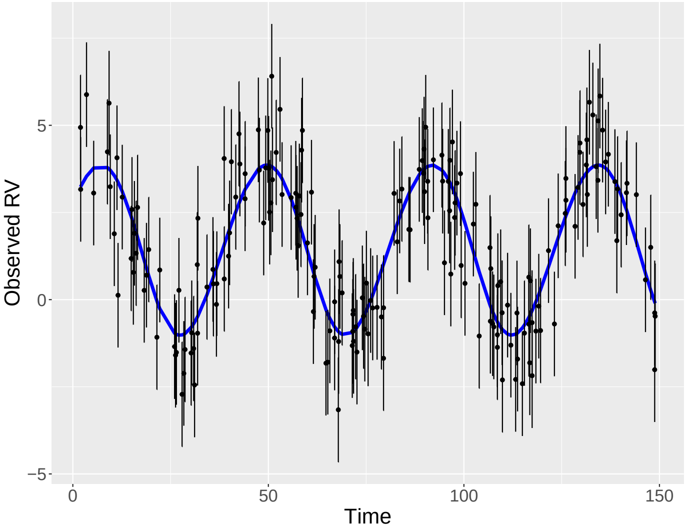
For $Y_1, \dots Y_n \overset{iid}{\sim} N(\mu, 1)$

- Write data generating equation as $Y_i = \mu + U_i$, $U_i \overset{iid}{\sim} N(0,1)$
- $\mathcal{L}(\mu) = \prod_{i=1}^{n} (2\pi)^{-1/2} \exp[-(y_i - \mu)^2/2]$
- $\nabla_\mu G(\mathbf{U}, \mu)|_{U_i = y_i - \mu} = [1, \dots, 1]'$, $J(\mathbf{Y}, \mu) = \sqrt{n}$
- $r_{\mathbf{Y}}(\mu) = (2\pi/n)^{-1/2} \exp\left(-\frac{(\mu - \bar{y})^2}{2(1/n)} \right)$

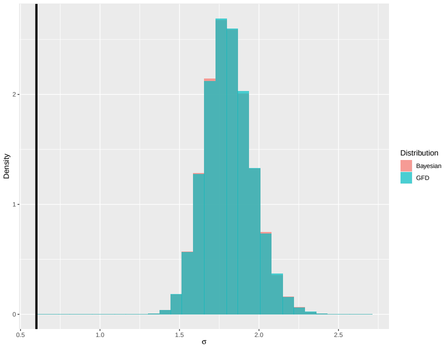
Simulations

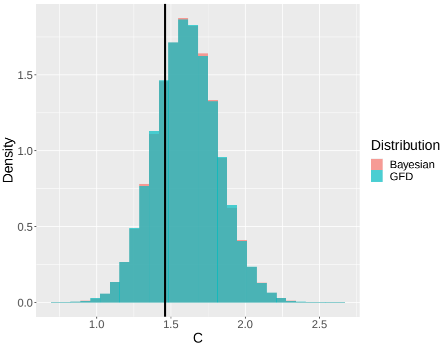
- Generate 200 observations from a one-planet system
- Use Markov Chain Monte Carlo with:
  - Flat priors to obtain Bayesian posterior samples (Nelson et al., 2020)
  - $J(\mathbf{Y}, \boldsymbol{\theta})$ as the joint prior distribution to obtain GFD samples
- Compare GFD samples to Bayesian posterior samples for 100 datasets

Data

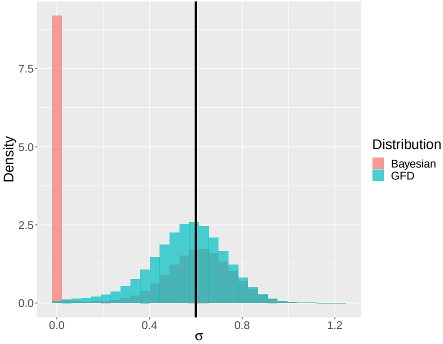


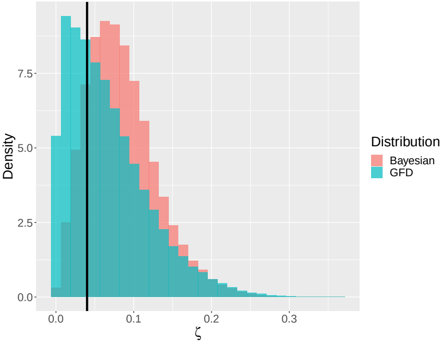
Fitting 0 Planet Model



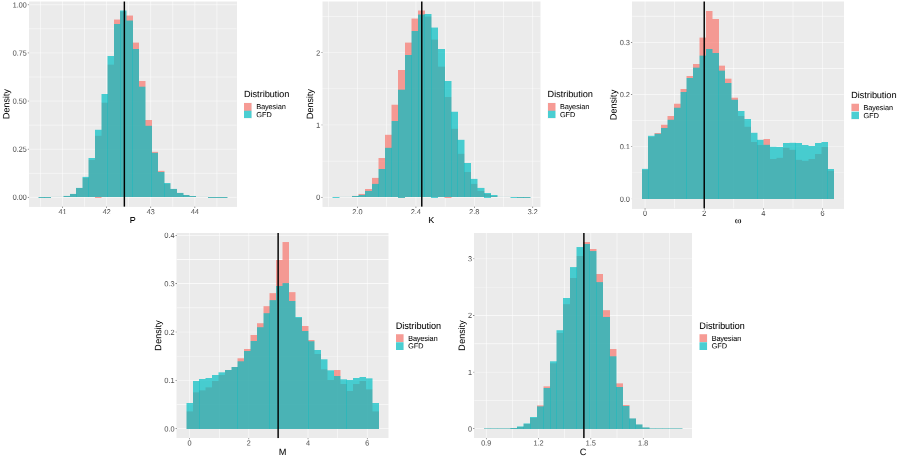


Fitting 1 Planet Model





Fitting 1 Planet Model



Model Selection using GFD

Compare two models S and S' using fiducial factor (Hannig et al., 2016):

$$FF := \frac{\int_{\mathbb{T}_S} \mathcal{L}(\boldsymbol{\vartheta}|S) J(\mathbf{Y}, \boldsymbol{\vartheta}|S) d\boldsymbol{\vartheta}}{\int_{\mathbb{T}_{S'}} \mathcal{L}(\boldsymbol{\vartheta}|S') J(\mathbf{Y}, \boldsymbol{\vartheta}|S') d\boldsymbol{\vartheta}}$$

Bayes Factor:

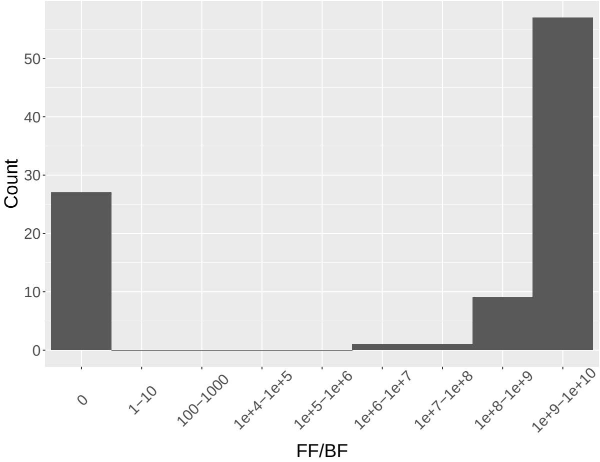
$$BF := \frac{\int_{\mathbb{T}_S} \mathcal{L}(\boldsymbol{\vartheta}|S) p(\boldsymbol{\vartheta}|S) d\boldsymbol{\vartheta}}{\int_{\mathbb{T}_{S'}} \mathcal{L}(\boldsymbol{\vartheta}|S') p(\boldsymbol{\vartheta}|S') d\boldsymbol{\vartheta}}$$

Ratio of FF to BF:

$$FF/BF = \frac{\int_{\mathbb{T}_S} \frac{J(\mathbf{Y}, \boldsymbol{\vartheta}|S)}{p(\boldsymbol{\vartheta}|S)} \Pi(\boldsymbol{\vartheta}|\mathbf{Y}, S) d\boldsymbol{\vartheta}}{\int_{\mathbb{T}_S} \frac{J(\mathbf{Y}, \boldsymbol{\vartheta}|S')}{p(\boldsymbol{\vartheta}|S')} \Pi(\boldsymbol{\vartheta}|\mathbf{Y}, S') d\boldsymbol{\vartheta}}$$

Ratio of FF to BF

- Peak at 0 due to Bayesian posterior estimates of σ equaling 0



Conclusions and Current Work

- Jacobian more accurately concentrates MCMC proposals around true parameters
- GFD computationally more stable
- Ongoing work:
  - Alternative parameterizations to improve fit
  - Initialize chains at MLE
  - Compare performance for data generated with 0, 2, and 3 planets
  - Additional ways to incorporate prior information

Thank you!

https://ngierty.github.io/

Jon Williams

Ryan Martin

References

Danby, J. 1992, Richmond: Willman-Bell

Ford, E. B., & Gregory, P. C. 2006, arXiv preprint astro-ph/0608328

Hannig, J., Iyer, H., Lai, R. C., & Lee, T. C. 2016, Journal of the American Statistical Association, 111, 1346

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Parviainen, H. 2017, arXiv preprint arXiv:1711.03329

Appendix

Notation

Observed Variables:

- $Y(t)$: Observed RV
- σ_t: Known noise
- t: Time stamp
- $\gamma(t)$: Spectrograph label

Unknown Parameters:

- σ: Observational noise
- ζ_i: *Eccentricity* of planet i's orbit
- C_j: *Velocity offset* for the jth spectrograph
- ω_i: *Argument of periapsis* for planet i
- P_i: *Orbital period* for planet i
- $M_{0,i}$: Position of planet i when it is closest to its parent star
- K_i: *Semi-amplitude* for planet i

Simulation Parameters

P (days)	K (m/s)	ζ (none)	ω (rad)	M (rad)	C (m/s)	σ (m/s)
42.4	2.44	0.04	2	2.99	1.46	0.6

Table 1: Nelson et al. (2020)

Nelson et al. (2020) Priors

Model Parameter	Variable	Prior Distribution	Minimum	Maximum
Amplitude of jitter	σ	$[(\sigma_0 + \sigma)\log(1 + \sigma_{\max}/\sigma_0)]^{-1}$	0	99 m/s
		Parameters for each velocity reference		
Velocity offset	C_j	$(2C_{\max})^{-1}$	-1000 m/s	1000 m/s
		Parameters for each planet		
Orbital Period	P_i	$[P\log(P_{\max}/P_{\min})]^{-1}$	1.25 days	10^4 days
Velocity semi-amplitude	K_i	$[(K_0 + K_i)\log(1 + K_{\max})/K_0]^{-1}$	0	999 m/s
Orbital eccentricity	ζ_i	$\frac{\zeta}{\sigma_\zeta^2}\exp(-\zeta^2/\sigma_\zeta^2)/[1 - exp(-1/(2\sigma_\zeta^2)]$	0	1
Argument of pariastron	ω_i	$(2\pi)^{-1}$	0	2π
Orbital Phase	$M_{0,i}$	$(2\pi)^{-1}$	0	2π

Regularity Conditions for GFD

- The function G has continuous partial derivatives with respect to all variables $\theta_j, j = 1, \ldots, p$ and $Z_i, i = 1, \ldots, n$.

- For each $\mathbf{y}$ and $\boldsymbol{\theta}$ there is at most one $\mathbf{z}$ so that $\mathbf{y} = G(\mathbf{z}, \boldsymbol{\theta})$. For the observed data $\mathbf{y}$ there is a $\boldsymbol{\theta}$ and $\mathbf{z}$ so that $\mathbf{y} = G(\mathbf{z}, \boldsymbol{\theta})$. Additionally, $\det\Big(\nabla_{\mathbf{z}} G(\mathbf{z}, \boldsymbol{\theta})\Big) \neq 0$ for all $\boldsymbol{\theta}, \mathbf{z}$

- $\nabla_{\boldsymbol{\theta}} G(\mathbf{z}, \boldsymbol{\theta})$ is full column-rank

- Entries of $\nabla_{\boldsymbol{\theta}} G(\mathbf{z}, \boldsymbol{\theta})$ has continuous partial derivatives with respect to all variables $\theta_j, j = 1, \ldots, p$ and $Z_i, i = 1, \ldots, n$